Introduction to Higher-degree Equations

Class: __________ Name: __________

Review: What We Learned Early About Equations

We start with the familiar equations from school:

- One-variable linear equation

$$ax + b = 0$$

- Two-variable linear system

$$\begin{cases} a_1 x + b_1 y = c_1 \\ a_2 x + b_2 y = c_2 \end{cases}$$

Solved by substitution or elimination.

- Three-variable linear system

Same idea: eliminate variables step by step to find a solution.

- One-variable quadratic equation

$$ax^2 + bx + c = 0$$

Up to this point, we mostly deal with linear equations and low-degree equations.

How This Leads to Linear Algebra

When the number of variables grows, for example:

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n = b_2 \\ \vdots \qquad \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n = b_m \end{cases}$$

the usual substitution/elimination becomes inefficient.

So mathematicians introduced matrices and vectors:

$$\mathbf{Ax} = \mathbf{b}$$

and developed systematic tools:

- Gaussian elimination
- determinants
- vector spaces

Linear algebra grew out of the need to understand and solve general n-variable linear systems in a clean and unified way.

Linear algebra = the theory behind multi-variable linear equations.

How This Leads to Abstract Algebra

On the other hand, consider higher-degree single-variable equations, e.g.

$$ax^3 + bx^2 + cx + d = 0$$
$$ax^4 + \cdots = 0$$

For cubics and quartics, formulas similar to the quadratic formula exist. But for the general quintic, mathematicians discovered there is no general solution by radicals.

To understand why, they began studying the symmetry of the roots. These permutations form a group, and this idea grew into the broader study of algebraic structures:

- groups
- rings
- fields

This became abstract algebra.

Abstract algebra = born from studying the symmetry structure of roots of 1-variable n-th degree equations.

We now turn to **higher-degree single-variable equations**; since general equations are too difficult to study directly, we focus on special cases such as those with **rational coefficients and roots**, and to handle them we will build a small toolbox: **polynomial division**, **Remainder Theorem, Factor Theorem**, **Rational Zero Theorem**, and **Vieta's formulas**.

Tool #1: Remainder and Factor Theorem

REMAINDER THEOREM

If the polynomial $P(x)$ is divided by $x - c$, then the remainder is the value $P(c)$.

Proof If the divisor in the Division Algorithm is of the form $x - c$ for some real number c, then the remainder must be a constant (since the degree of the remainder is less than the degree of the divisor). If we call this constant r, then

$$P(x) = (x - c) \cdot Q(x) + r$$

Replacing x by c in this equation, we get $P(c) = (c - c) \cdot Q(c) + r = 0 + r = r$, that is, $P(c)$ is the remainder r. ■

FACTOR THEOREM

c is a zero of P if and only if $x - c$ is a factor of $P(x)$.

Proof If $P(x)$ factors as $P(x) = (x - c)Q(x)$, then

$$P(c) = (c - c)Q(c) = 0 \cdot Q(c) = 0$$

Conversely, if $P(c) = 0$, then by the Remainder Theorem

$$P(x) = (x - c)Q(x) + 0 = (x - c)Q(x)$$

so $x - c$ is a factor of $P(x)$. ■

☐ Practice 1

Let $P(x) = x^3 - 7x + 6$. Show that $P(1) = 0$, and use this fact to factor $P(x)$ completely.

$$x - 1\overline{)x^3 + 0x^2 - 7x + 6}$$

☐ Practice 2

Find a polynomial of degree four that has zeros -3, 0, 1, and 5, and the coefficient of x^3 is -6.

Tool #2: Rational Zero Theorem

We consider polynomial equations with **rational coefficients**, which is essentially the same as **integer-coefficient** polynomials for the purpose of finding roots.

$$\frac{3}{5}x^2 - \frac{7}{2}x + 1 = 0$$

$$6x^3 - 35x + 10 = 0$$

RATIONAL ZEROS THEOREM

If the polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has integer coefficients (where $a_n \neq 0$ and $a_0 \neq 0$), then every rational zero of P is of the form

$$\frac{p}{q}$$

where p and q are integers and

p is a factor of the constant coefficient a_0

q is a factor of the leading coefficient a_n

Proof If p/q is a rational zero, in lowest terms, of the polynomial P, then we have

$$a_n\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \cdots + a_1\left(\frac{p}{q}\right) + a_0 = 0$$

$$a_n p^n + a_{n-1}p^{n-1}q + \cdots + a_1 pq^{n-1} + a_0 q^n = 0 \quad \text{Multiply by } q^n$$

$$p(a_n p^{n-1} + a_{n-1}p^{n-2}q + \cdots + a_1 q^{n-1}) = -a_0 q^n \quad \text{Subtract } a_0q^n \text{ and factor LHS}$$

Now p is a factor of the left side, so it must be a factor of the right side as well. Since p/q is in lowest terms, p and q have no factor in common, so p must be a factor of a_0. A similar proof shows that q is a factor of a_n. ■

☐ Practice 3

Prove that q is a factor of a_n.

☐ Example

Find the rational zeros of $P(x) = x^3 - 3x + 2$.

SOLUTION Since the leading coefficient is 1, any rational zero must be a divisor of the constant term 2. So the possible rational zeros are ± 1 and ± 2. We test each of these possibilities.

$$P(1) = (1)^3 - 3(1) + 2 = 0$$

$$P(-1) = (-1)^3 - 3(-1) + 2 = 4$$

$$P(2) = (2)^3 - 3(2) + 2 = 4$$

$$P(-2) = (-2)^3 - 3(-2) + 2 = 0$$

The rational zeros of P are 1 and -2.

☐ Practice 4

$2x^3 + x^2 - 13x + 6 = 0$

Hint: Use Rational Zero Theorem to find a root -> polynomial division

☐ Practice 5

$x^4 - 5x^3 - 5x^2 + 23x + 10 = 0$

Hint: Use Rational Zero Theorem to find **a root** -> polynomial division -> Use Rational Zero Theorem to find **another root** -> polynomial division -> quadratic formula to get **the last two roots**

Tool #3: Vieta's formulas

We have learned Vieta's formulas for quadratic equations, originally proved using the quadratic formula. For higher-degree equations, root formulas are much more complicated, but we can still derive Vieta's formulas without using these formulas, simply by applying factorization.

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = a_n(x - r_1)(x - r_2)\cdots(x - r_n)$$

Expanding the products in the right-hand side, and equating the coefficients of each power of x between the two members of the equation.

$$\begin{aligned}
r_1 + r_2 + r_3 + \cdots + r_n &= -\frac{a_{n-1}}{a_n} \\
r_1r_2 + r_1r_3 + r_1r_4 + \cdots + r_{n-2}r_{n-1} &= \frac{a_{n-2}}{a_n} \\
r_1r_2r_3 + r_1r_2r_4 + \cdots + r_{n-2}r_{n-1}r_n &= -\frac{a_{n-3}}{a_n} \\
&\vdots \\
r_1r_2r_3\cdots r_n &= (-1)^n\frac{a_0}{a_n}.
\end{aligned}$$

Vieta's Formulae for Cubics. Suppose the cubic $ax^3 + bx^2 + cx + d$ has roots r_1, r_2, and r_3. Then

$$r_1 + r_2 + r_3 = \frac{-b}{a}$$

$$r_1r_2 + r_1r_3 + r_2r_3 = \frac{c}{a}$$

$$r_1r_2r_3 = \frac{-d}{a}.$$

☐ Practice 6

Let $g(x) = x^3 - 5x^2 + 2x - 7$, and let the roots of $g(x)$ be $p, q,$ and r. Compute $p^2qr + pq^2r + pqr^2$.

☐ Practice 7

The roots r_1, r_2, r_3 of $x^3 - 2x^2 - 11x + a = 0$ satisfy $r_1 + 2r_2 + 3r_3 = 0$. Find all possible values of a.

☐ Practice 8

Suppose that the roots of $x^3 + 3x^2 + 4x - 11 = 0$ are a, b, and c, and that the roots of $x^3 + rx^2 + sx + t = 0$ are a + b, b + c, and c + a. Find t.